REIDEMEISTER TORSION, TWISTED ALEXANDER POLYNOMIAL AND FIBERED KNOTS

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Abstract. As a generalization of a classical result on the Alexander polynomial for fibered knots, we show in this paper that the Reidemeister torsion associated to a certain representation detects fiberedness of knots in the three sphere.

1. Introduction

As is well-known, the Alexander polynomial of a fibered knot is monic (see [13], [14], [16]). That is, the coefficient of the highest degree term of the normalized Alexander polynomial is a unit $1 \in \mathbb{Z}$. By the symmetry (or duality) of the Alexander polynomial, its lowest degree term is also one. This criterion is sufficient for alternating knots [12] and prime knots up to 10 crossings [4] for instance. However, in general, the converse is not true. In fact, there are infinitely many non-fibered knots having monic Alexander polynomials. If we remember here Milnor's result [9], we have to remark that these claims on the Alexander polynomial can be restated by the Reidemeister torsion.

The purpose of this paper is to give a necessary condition that a knot in S^3 is fibered by virtue of the Reidemeister torsion associated to a certain linear representation. More precisely, we show that the Reidemeister torsion of a fibered knot defined for a certain tensor representation is expressed as a rational function of monic polynomials. This Reidemeister torsion is nothing but Wada's twisted Alexander polynomial (see [7] for details), so that our result can be regarded as a natural generalization of the property on the classical Alexander polynomial mentioned above.

This paper is organized as follows. In the next section, we review the definition of Reidemeister torsion over a field $\mathbb{F}$. Further we describe how to compute it in the case of knot exteriors. The point of our method here is that the notion of monic makes sense for the Reidemeister torsion associated to a tensor representation of a unimodular representation over $\mathbb{F}$ and the abelianization homomorphism. In Section 3, we state and prove the main theorem of this paper. The final section is devoted to compute some examples.

We should note here that there is a similar work by Cha [1]. The notion of Alexander polynomials twisted by a representation and its applications have appeared in several papers (see [2], [6], [8], [18]).

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2. Reidemeister torsion

In this section, we review the definition of Reidemeister torsion over a field $\mathbb{F}$ (see [3] and [10] for details).

Let V be an n-dimensional vector space over $\mathbb{F}$, and $\mathbf{b} = (b_1, \dots, b_n)$ and $\mathbf{c} = (c_1, \dots, c_n)$ two bases for V. If we put $c_i = \sum_{j=1}^{n} a_{ij} b_j$, we have a nonsingular matrix $A = (a_{ij})$ with coefficients in $\mathbb{F}$. Further let $[\mathbf{b}/\mathbf{c}]$ denote the determinant of A.

Now let us consider an acyclic chain complex of finite dimensional vector spaces over $\mathbb{F}$:

$$C_* : 0 \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0.$$

We assume that a preferred basis $\mathbf{c}_q$ for $C_q(C_*)$ is given for any q. Choose any basis $\mathbf{b}_q$ of $B_q(C_*)$ and take its lift in $C_{q+1}(C_*)$, which we denote by $\widetilde{\mathbf{b}}_q$.

Since the natural inclusion map

$$B_q(C_*) \to Z_q(C_*)$$

is an isomorphism, the basis $\mathbf{b}_q$ can serve as a basis for $Z_q(C_*)$. Similarly the sequence

$$0 \longrightarrow Z_q(C_*) \longrightarrow C_q(C_*) \longrightarrow B_{q-1}(C_*) \longrightarrow 0$$

is exact and the vectors $(\mathbf{b}_q, \widetilde{\mathbf{b}}_{q-1})$ is a basis for $C_q(C_*)$. It is easily shown that $[\mathbf{b}_q, \widetilde{\mathbf{b}}_{q-1}/\mathbf{c}_q]$ is independent of the choices of $\widetilde{\mathbf{b}}_{q-1}$. Hence we may simply denote it by $[\mathbf{b}_q, \mathbf{b}_{q-1}/\mathbf{c}_q]$.

Definition 2.1. The torsion of the chain complex C_* is defined by the alternating product

$$\prod_{q=0}^{m} [\mathbf{b}_q, \mathbf{b}_{q-1}/\mathbf{c}_q]^{(-1)^{q+1}}$$

and we denote it by $\tau(C_*)$.

Remark 2.2. The torsion $\tau(C_*)$ depends only on the bases $\mathbf{c}_0, \dots, \mathbf{c}_m$.

Now let us apply the above torsion to the following geometric situations. Let X be a finite cell complex and $\widetilde{X}$ the universal covering of X with the right action of $\pi_1 X$ as deck transformations. Then the chain complex $C_*(\widetilde{X}, \mathbb{Z})$ has a structure of right free $\mathbb{Z}[\pi_1 X]$-modules. Let

$$\rho : \pi_1 X \to GL(n, \mathbb{F})$$

be a linear representation. We may regard V as a $\pi_1 X$-module by using ρ and denote it by V_ρ. Define the chain complex $C_*(X, V_\rho)$ by $C_*(\widetilde{X}, \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1 X]} V_\rho$ and choose a preferred basis

$$\{\sigma_1 \otimes e_1, \sigma_1 \otimes e_2, \dots, \sigma_1 \otimes e_n, \dots, \sigma_{k_q} \otimes e_1, \dots, \sigma_{k_q} \otimes e_n\}$$

of $C_q(X, V_\rho)$, where $\{e_1, \dots, e_n\}$ is a basis of V and $\sigma_1, \dots, \sigma_{k_q}$ are q-cells giving the preferred basis of $C_q(\widetilde{X}, \mathbb{Z})$.

Now we consider the following situation. That is, $C_*(X, V_\rho)$ is acyclic, in other words, all homology groups $H_*(X, V_\rho)$ vanish. In this case, we call ρ an acyclic representation.

Definition 2.3. Let $\rho : \pi_1 X \to GL(n, \mathbb{F})$ be an acyclic representation. Then Reidemeister torsion of X with V_ρ-coefficients is defined by the torsion of the chain complex $C_*(X, V_\rho)$. We denote it by $\tau_\rho(X)$.

Remark 2.4. It is known that $\tau_\rho(X)$ is well-defined as a PL-invariant, for an acyclic representation $\rho : \pi_1 X \to GL(n, \mathbb{F})$, up to a factor $\pm d$ where $d \in \mathrm{Im}(\det \circ \rho) \subset \mathbb{F}^*$. As a reference, see [10] Section 8. We can easily make a refinement of the above argument for our situation.

Here let us consider a knot K in S^3 and its exterior E. For the knot group $\pi K = \pi_1 E$, we choose and fix a Wirtinger presentation

$$P(\pi K) = \langle x_1, \dots, x_u \mid r_1, \dots, r_{u-1} \rangle.$$

Then we can construct a 2-dimensional cell complex X from $P(\pi K)$ such that E collapses to X. The abelianization homomorphism

$$\alpha : \pi K \to H_1(E, \mathbb{Z}) \cong \mathbb{Z} = \langle t \rangle$$

is given by

$$\alpha(x_1) = \cdots = \alpha(x_u) = t.$$

Furthermore, we always suppose that the image of a representation $\rho : \pi K \to GL(n, \mathbb{F})$ is included in $SL(n, \mathbb{F})$.

These maps naturally induce the ring homomorphisms $\tilde{\rho}$ and $\tilde{\alpha}$ from $\mathbb{Z}[\pi K]$ to $M(n, \mathbb{F})$ and $\mathbb{Z}[t^{\pm 1}]$ respectively, where $M(n, \mathbb{F})$ denotes the matrix algebra of degree n over $\mathbb{F}$. Then $\tilde{\rho} \otimes \tilde{\alpha}$ defines a ring homomorphism

$$\mathbb{Z}[\pi K] \to M\left(n, \mathbb{F}[t^{\pm 1}]\right).$$

Let F_u denote the free group with generators $x_1, \dots, x_u$ and denote by

$$\Phi : \mathbb{Z}[F_u] \to M\left(n, \mathbb{F}[t^{\pm 1}]\right)$$

the composite of the surjection $\mathbb{Z}[F_u] \to \mathbb{Z}[\pi K]$ induced by the presentation and the map $\mathbb{Z}[\pi K] \to M(n, \mathbb{F}[t^{\pm 1}])$ given by $\tilde{\rho} \otimes \tilde{\alpha}$.

Let us consider the $(u-1) \times u$ matrix M whose (i, j) component is the $n \times n$ matrix

$$\Phi\left(\frac{\partial r_i}{\partial x_j}\right) \in M\left(n, \mathbb{F}[t^{\pm 1}]\right),$$

where $\dfrac{\partial}{\partial x}$ denotes the free differential calculus. This matrix M is called the Alexander matrix of the presentation $P(\pi K)$ associated to the representation ρ.

For $1 \leq j \leq u$, let us denote by M_j the $(u-1) \times (u-1)$ matrix obtained from M by removing the jth column. We regard M_j as a $(u-1)n \times (u-1)n$ matrix with coefficients in $\mathbb{F}[t^{\pm 1}]$.

Now let us recall that the tensor representation

$$\rho \otimes \alpha : \pi K \to GL(n, \mathbb{F}(t))$$

is defined by $(\rho \otimes \alpha)(x) = \rho(x)\alpha(x)$ for $x \in \pi K$. Here $\mathbb{F}(t)$ denotes the rational function field over $\mathbb{F}$ and let V be the n-dimensional vector space over $\mathbb{F}(t)$. Hereafter, we denote the Reidemeister torsion $\tau_{\rho\otimes\alpha}(E)$ by $\tau_{\rho\otimes\alpha} K$.

Theorem 2.5. *All homology groups* $H_*(E, V_{\rho\otimes\alpha})$ *vanish* (*namely,* $\rho\otimes\alpha$ *is an acyclic representation*) *if and only if* $\det M_j \neq 0$ *for some* j*. In this case, we have*

$$\tau_{\rho\otimes\alpha}K = \frac{\det M_j}{\det\Phi(x_j - 1)},$$

for any j $(1 \leq j \leq u)$*. Moreover,* $\tau_{\rho\otimes\alpha}$ *is well-defined up to a factor* $\pm t^{nk}$ $(k \in \mathbb{Z})$ *if* n *is odd and up to only* t^{nk} *if* n *is even.*

Proof. The first two assertion are nothing but [7] Proposition 3.1. The independence on j follows from [7] Lemma 1.2.

Next, if we consider well-definedness up to $\pm t^{nk}$ $(k \in \mathbb{Z})$, we have only to recall Remark 2.4. The image of

$$\det\circ(\rho\otimes\alpha) : \pi K \to GL(n, \mathbb{F}(t)) \to \mathbb{F}(t)^*$$

is just $\{t^{nk} | k \in \mathbb{Z}\}$, because Im ρ is included in $SL(n, \mathbb{F})$. Therefore the claim immediately follows. Here if we take an even dimensional unimodular representation, $\tau_{\rho\otimes\alpha}$ is well-defined up to t^{nk} (see also [3] for details). $\square$

Remark 2.6. This theorem asserts that the twisted Alexander polynomial [18] of a knot is the Reidemeister torsion of its knot exterior (see [7] for details). This is a generalization of Milnor's theorem in [9]. Recently this framework extended to more general situations by Kirk-Livingston in [6].

Remark 2.7. Assume that ρ is a homomorphism to $SL(n, R)$ over a unique factorization domain R and the knot group πK has a presentation which is strongly Tietze equivalent to a Wirtinger presentation of the knot. Then Wada shows in [18] that the twisted Alexander polynomial of the knot associated to ρ is well-defined up to a factor $\pm t^{nk}$ $(k \in \mathbb{Z})$ if n is odd and up to only t^{nk} if n is even.

Remark 2.8. If there is an element γ of the commutator subgroup of πK such that 1 is not an eigenvalue of $\rho(\gamma)$, then $\tau_{\rho\otimes\alpha}$ becomes a "polynomial" (see [18]). Namely $\det M_j$ is divided by $\det\Phi(x_j - 1)$.

3. Main theorem

In this section, we give a necessary condition that a knot K in S^3 is fibered. A polynomial $a_m t^m + \cdots + a_1 t + a_0 \in \mathbb{F}[t]$ is called monic if the coefficient a_m is one. We then see from Theorem 2.5 that the notion of monic polynomial makes sense for the Reidemeister torsion.

Theorem 3.1. *For a fibered knot* K *in* S^3 *and a unimodular representation* $\rho : \pi K \to SL(2n, \mathbb{F})$*, the Reidemeister torsion* $\tau_{\rho\otimes\alpha}K$ *is expressed as a rational function of monic polynomials.*

Proof. By using the fiber bundle structure, we can take the following presentation of πK:

$$P(\pi K) = \langle x_1, \ldots, x_{2g}, h \mid r_i = h x_i h^{-1} \varphi_*(x_i)^{-1},\ 1 \leq i \leq 2g\rangle,$$

where $x_1, \ldots, x_{2g}$ is a generating system of the fundamental group of the fiber surface of genus g, h is a generator for S^1-direction corresponding to the meridian of K and φ_* denotes the automorphism of the surface group induced by the monodromy map φ. Here the abelianization homomorphism $\alpha : \pi K \to \mathbb{Z} = \langle t\rangle$ is given by

$$\alpha(x_1) = \cdots = \alpha(x_{2g}) = 1 \quad \text{and} \quad \alpha(h) = t.$$

This presentation of πK allows us to define another 2-dimensional cell complex Y instead of a cell complex X constructed from a Wirtinger presentation of πK. Namely it has only one vertex, $2g+1$ edges, and $2g$ 2-cells attached by the relations of $P(\pi K)$. It is easy to see that there exists a homotopy equivalence $f : E \to Y$. From the result of Waldhausen [19], the Whitehead group $Wh(\pi K)$ is trivial for a knot group in general. Thus the Whitehead torsion of f is also trivial element in $Wh(\pi K)$. It then implies that the homotopy equivalence map f induces a simple homotopy equivalence from E to Y. Since the Reidemeister torsion is a simple homotopy invariant, we can compute the Reidemeister torsion of E as the one of Y as follows. That is, we may use the previous presentation $P(\pi K)$ to compute $\tau_{\rho\otimes\alpha}K$ by means of Theorem 2.5.

Let us consider the "big" $2g \times 2g$ matrix M whose (i, j) component is the $n \times n$ matrix

$$\Phi\left(\frac{\partial r_i}{\partial x_j}\right) \in M(n, \mathbb{F}[t^{\pm 1}]).$$

We then see that the diagonal component of M is

$$\begin{aligned}\Phi\left(\frac{\partial r_i}{\partial x_i}\right) &= \Phi\left(h - \frac{\partial \varphi_*(x_i)}{\partial x_i}\right)\\ &= t\rho(h) - \tilde{\rho}\left(\frac{\partial \varphi_*(x_i)}{\partial x_i}\right)\end{aligned}$$

and the coefficient of the highest degree term of $\det\Phi(\partial r_i/\partial x_i)$ is just $\det\,\rho(h) = 1$. Further other components $\Phi(\partial r_i/\partial x_j)$ $(i \neq j)$ do not contain t, so that the coefficient of the highest degree term of $\det M$ is also one.

On the other hand, the denominator is given by

$$\begin{aligned}\det\Phi(h-1) &= \det(t\rho(h) - I)\\ &= (\det\,\rho(h))t^{2n} - (\mathrm{tr}\,\rho(h))t^{2n-1} + \cdots + 1\\ &= t^{2n} + \cdots + 1,\end{aligned}$$

where I denotes the identity matrix. Moreover ρ is an even dimensional representation, so we see that $\tau_{\rho\otimes\alpha}K$ is well-defined up to a factor t^{2nk} $(k \in \mathbb{Z})$. This completes the proof. □

Remark 3.2. If we can show directly that the presentation $P(\pi K)$ in Theorem 3.1 is strongly Tietze equivalent to a Wirtinger presentation of K, then Theorem 3.1 follows without using the result of Waldhausen (see Remark 2.7).

Remark 3.3. If $\mathbb{F}$ is $\mathbb{C}$ or a finite field $\mathbb{F}_p$, then the Reidemeister torsion $\tau_{\rho\otimes\alpha}K$ for any knot K and any representation $\rho : \pi K \to SL(2, \mathbb{F})$ is symmetric. Namely, $\tau_{\rho\otimes\alpha}K$ is invariant under the transformation $t \mapsto t^{-1}$ up to a factor t^k $(k \in \mathbb{Z})$. Such a duality theorem appears originally in [9]. See also [6], [7] and [11] as for related works.

4. Examples

Example 4.1. Let K be the figure eight knot 4_1. This is one of the well-known genus one fibered knots in S^3. The fundamental group of the exterior has a presentation

$$\pi K = \langle x, y \mid zxz^{-1}y^{-1}\rangle,$$

where $z = x^{-1}yxy^{-1}x^{-1}$. Let $\rho : \pi K \to SL(2,\mathbb{C})$ be a noncommutative representation defined by

$$\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix},$$

where ω is a complex number satisfying $\omega^2 + \omega + 1 = 0$. As pointed out by Wada in [18], it is convenient to use relations instead of relators for the computation of the Alexander matrix. Thereby applying free differential calculus to the relation $r : zx = yz$, we obtain

$$\frac{\partial r}{\partial x} = -x^{-1} + x^{-1}y + yx^{-1} - yx^{-1}y + yx^{-1}yxy^{-1}x^{-1}.$$

Thus we have the matrix

$$M_2 = \left(\Phi\left(\frac{\partial r}{\partial x}\right)\right) = \begin{pmatrix} -(\omega+1)t + \omega + 2 - t^{-1} & t + \omega - 2 + t^{-1} \\ (\omega-1)t - \omega + 1 & -(\omega+1)t + 3 - t^{-1} \end{pmatrix}.$$

Then the numerator of $\tau_{\rho\otimes\alpha}$ is given by

$$\begin{aligned} \det M_2 &= t^{-2}(t^4 - 6t^3 + \omega^4 t^2 + \omega^2 t^2 + 11t^2 - 6t + 1) \\ &= t^{-2}(t-1)^2(t^2 - 4t + 1). \end{aligned}$$

On the other hand, the denominator of $\tau_{\rho\otimes\alpha}$ is given by

$$\begin{aligned} \det\Phi(y-1) &= \det(t\rho(y) - I) \\ &= \det\begin{pmatrix} t-1 & 0 \\ -\omega t & t-1 \end{pmatrix} \\ &= (t-1)^2. \end{aligned}$$

Therefore the Reidemeister torsion of the figure eight knot K is

$$\tau_{\rho\otimes\alpha}K = t^2 - 4t + 1$$

and this is in fact a monic polynomial.

Example 4.2. Let KT be the Kinoshita-Terasaka knot [5]. It is well-known that KT is one of the classical examples of knots with the trivial Alexander polynomial. The knot group πKT has a presentation with four generators $x_1, \ldots, x_4$ and three relations [18]:

$$\begin{aligned} r_1 &: x_1x_2x_1^{-1} = x_4x_2x_4x_2^{-1}x_4^{-1}, \\ r_2 &: x_4x_2x_4^{-1} = x_2^{-1}x_3x_1x_3^{-1}x_2x_1x_2^{-1}x_3x_1^{-1}x_3^{-1}x_2, \\ r_3 &: x_1x_3x_1^{-1} = x_4x_3x_4x_3^{-1}x_4^{-1}. \end{aligned}$$

Applying free differential calculus, we have

$$\frac{\partial r_1}{\partial x_1} = 1 - x_1x_2x_1^{-1}, \qquad \frac{\partial r_1}{\partial x_2} = x_1 - x_4 + x_4x_2x_4x_2^{-1}, \qquad \frac{\partial r_1}{\partial x_3} = 0,$$

$$\begin{aligned}
\frac{\partial r_2}{\partial x_1} &= -x_2^{-1}x_3 - x_2^{-1}x_3x_1x_3^{-1}x_2 + x_2^{-1}x_3x_1x_3^{-1}x_2x_1x_2^{-1}x_3x_1^{-1},\\
\frac{\partial r_2}{\partial x_2} &= x_4 + x_2^{-1} - x_2^{-1}x_3x_1x_3^{-1} + x_2^{-1}x_3x_1x_3^{-1}x_2x_1x_2^{-1}\\
&\quad - x_2^{-1}x_3x_1x_3^{-1}x_2x_1x_2^{-1}x_3x_1^{-1}x_3^{-1},\\
\frac{\partial r_2}{\partial x_3} &= -x_2^{-1} + x_2^{-1}x_3x_1x_3^{-1} - x_2^{-1}x_3x_1x_3^{-1}x_2x_1x_2^{-1}\\
&\quad + x_2^{-1}x_3x_1x_3^{-1}x_2x_1x_2^{-1}x_3x_1^{-1}x_3^{-1},
\end{aligned}$$

$$\frac{\partial r_3}{\partial x_1} = 1 - x_1x_3x_1^{-1}, \qquad \frac{\partial r_3}{\partial x_2} = 0, \qquad \frac{\partial r_3}{\partial x_3} = x_1 - x_4 + x_4x_3x_4x_3^{-1}.$$

Let $\rho : \pi KT \to SL(2, \mathbb{F}_5)$ be a noncommutative representation over the finite field $\mathbb{F}_5$ defined as follows:

$$\rho(x_1) = \begin{pmatrix} 0 & 1 \\ 4 & 1 \end{pmatrix}, \quad \rho(x_2) = \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix}, \quad \rho(x_3) = \begin{pmatrix} 0 & 1 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad \rho(x_4) = \begin{pmatrix} 4 & 4 \\ 3 & 2 \end{pmatrix}.$$

Then we obtain

$$M_4 = \begin{pmatrix}
3t+1 & t & t & t^2+2t & 0 & 0\\
2t & t+1 & 4t^2+t & 4t & 0 & 0\\
1 & 4t+3 & t+2+t^{-1} & t+t^{-1} & 3t+3+4t^{-1} & 3t+4t^{-1}\\
4t & 2t+1 & t+1+4t^{-1} & 3t+4 & 2t+4+t^{-1} & 4t+1\\
1 & 4t & 0 & 0 & 3t^2+t & 2t^2+2t\\
t & 4t+1 & 0 & 0 & 4t^2+t & 3t^2+4t
\end{pmatrix}.$$

Therefore the Reidemeister torsion of KT is given by

$$\begin{aligned}
\tau_{\rho\otimes\alpha} KT &= \frac{\det\, M_4}{\det\Phi(x_4-1)}\\
&= \frac{t^2(4t^8+t^7+t^6+4t^5+3t^4+4t^3+t^2+t+4)}{t^2+4t+1}\\
&= 4t^6+2t^4+t^3+2t^2+4.
\end{aligned}$$

This is well-defined up to a factor t^{2k} ($k \in \mathbb{Z}$), so that we may conclude the Kinoshita-Terasaka knot KT is not fibered.

Example 4.3. Let K be the knot illustrated in Figure 1. The normalized Alexander polynomial of K is equal to the monic polynomial $t^4 - t^3 + t^2 - t + 1$. The knot group πK has a presentation with seven generators $x_1, \ldots, x_7$ and six relations:

$$\begin{aligned}
&r_1 : x_2x_1 = x_3x_2x_1x_2x_1^{-1}x_2^{-1},\\
&r_2 : x_6x_5x_6^{-1} = x_4x_3x_1^{-1}x_3x_1^{-1}x_3x_1x_3^{-1}x_1x_3^{-1}x_1x_3^{-1}x_4^{-1},\\
&r_3 : x_6x_7x_6^{-1} = x_4x_3x_1^{-1}x_3x_1^{-1}x_3x_1x_3^{-1}x_1x_3^{-1}x_4^{-1},\\
&r_4 : x_5x_6x_5^{-1} = x_7x_2x_7^{-1},\\
&r_5 : x_2x_6x_2^{-1} = x_3x_2x_1x_2x_1^{-1}x_2^{-1}x_3^{-1}x_7x_3x_2x_1x_2^{-1}x_1^{-1}x_2^{-1}x_3^{-1},\\
&r_6 : x_5x_4x_5^{-1}x_7 = x_7x_3x_2x_1x_2x_1^{-1}x_2^{-1}x_3^{-1}.
\end{aligned}$$

Let $\rho : \pi K \to SL(2, \mathbb{F}_5)$ be a noncommutative representation over $\mathbb{F}_5$ defined as follows:

$$\rho(x_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(x_2) = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \quad \rho(x_3) = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \quad \rho(x_4) = \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix},$$

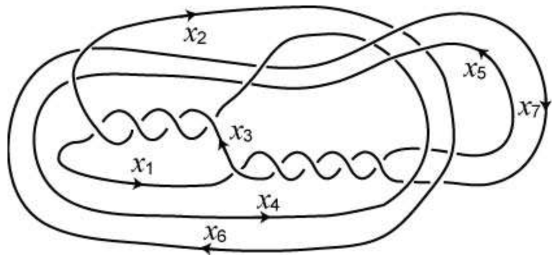

FIGURE 1.

$$\rho(x_5) = \begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix}, \quad \rho(x_6) = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad \rho(x_7) = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}.$$

By the same method as in previous examples, we have the following Reidemeister torsion of K:

$$\begin{aligned} \tau_{\rho\otimes\alpha} K &= \frac{\det\, M_7}{\det \Phi(x_7 - 1)} \\ &= \frac{t^{12}(3t^4 + 4t^3 + t^2 + 4t + 3)}{t^2 + 3t + 1} \\ &= 3t^2 + 3. \end{aligned}$$

Hence this knot K is not fibered.

We use Kodama's program "KNOT" and Wada's one to compute these examples. The former is to obtain N-data (see [17]) from a knot projection, which are necessary to input into Wada's program. The latter one is to compute unimodular representations over finite fields of knot groups from N-data. Here we should remark that Kodama's one works on Linux while Wada's one works on Macintosh.

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